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The principle of strictly increasing mixing character in incompatible general measurement

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Abstract. It is known that the density operators ρ of a quantal system are grouped into classes of equally mixed ones. The class to which ρ belongs is its so-called mixing character. The set of classes is known to be a lattice. The Ruch and Mead principle of increasing mixing character in complete measurement is extended to general observables. It is shown that the principle of strictly increasing mixing character holds in general incompatible measurement, and some consequences are discussed. A sufficient condition for strictly mixing-homomorphic functionals, i.e. for those which preserve the 'strictly larger' relation in the lattice, is obtained. Density-operator-dependent strengthening of the natural pre-order in the set of general observables is investigated.

1. Introduction

Uhlmann (1972 and 1973) introduced a quasi-order (or pre-order) binary relation '$\succ$' called 'more mixed' between some density operators:

$$\rho' \succ \rho \qquad \text{if} \qquad \rho' = \sum_\lambda c_\lambda U^{(\lambda)} \rho U^{(\lambda)+},$$

where the $U^{(\lambda)}$ are unitary, $U^{(\lambda)+}$ are adjoint, and $c_\lambda > 0$, $\Sigma_\lambda c_\lambda = 1$. Actually the second relation has been used already in a paper by Jaynes (1957b) in a somewhat different context. Such a relation is reflexive and transitive turning the set of density operators into a so-called quoset (short for: quasi-ordered set). It induces an equivalence relation:

$$\rho' \sim \rho \qquad \text{if} \qquad (\rho' \succ \rho) \qquad \text{and} \qquad (\rho \succ \rho'),$$

and it makes the quotient set of the quoset partially ordered (through the 'more mixed' relation of class representatives), i.e., it turns it into a so-called poset (short for: partially ordered set). Wehrl (1974) proved that it has the structure of a lattice.

This poset was rediscovered in another context by Ruch (1975). He introduced the term 'mixing character' of ρ for Uhlmann's class to which ρ belongs, and denoted it by $m[\rho]$ (cf also Lesche 1976).

We designate (for reasons that will become apparent) Ruch's 'larger mixing character' by '$\geqslant$'. One has $\rho' \sim \rho$ if and only if ρ' and ρ have the same positive eigenvalues with the same multiplicities. Writing the eigenvalues of ρ' as $p'_1 \geqslant p'_2 \geqslant \ldots$

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and those of ρ as $p_1 \geqslant p_2 \geqslant \ldots$, one has

$$m[\rho'] \geqslant m[\rho] \Leftrightarrow \forall s, \qquad \sum_{j=1}^{s} p'_j \leqslant \sum_{j=1}^{s} p_j. \tag{1}$$

We say that $m[\rho']$ is *strictly larger* than $m[\rho]$ and write $m[\rho'] > m[\rho]$ if $m[\rho'] \geqslant m[\rho]$ and $m[\rho'] \neq m[\rho]$. We show in this note that it is rather important to make this apparently trivial distinction of 'strictly larger' and 'equal' within '$\geqslant$'.

In an N-dimensional state space, there exists a special density operator $(1/N)I$ (I being the identity operator) describing the 'most mixed' ensemble. In contrast to this, every state $|\psi\rangle\langle\psi|$ actually contains no 'mixing' at all. It is easily seen that (1) implies

$$m[(1/N)I] > m[\rho] > m[|\psi\rangle\langle\psi|] \tag{2}$$

for every mixed density operator $\rho(\rho^2 \neq \rho)$ that differs from $(1/N)I$. The second inequality in (2) holds also in infinitely dimensional state spaces.

Ruch (1975) showed that in the classical discrete approximation of statistical mechanics (cf Wehrl 1978) when a master equation governs the dynamics, the principle of increasing mixing character is valid. It is easily seen that one may replace '$\geqslant$' by '$>$' in this monotonic increase. Subsequently Ruch and Mead (1976) showed that the principle is true also in the measurement of any complete quantal observable.

Since the possibility that the second law of thermodynamics might be strengthened by substituting mixing character for entropy is opened up, we feel that an investigation aimed to clarify the basic relations between mixing character and entropy is physically relevant.

In § 2 the existence of a natural and absolute quoset structure in the set of general (incomplete and complete) observables with purely discrete spectra is pointed out, and the Ruch-and-Mead principle is extended to the measurement of incomplete observables. Calling 'incompatible measurement' that of an observable incompatible (i.e. non-commuting) with the density operator of the ensemble measured, the principle of strictly increasing mixing character in incompatible measurement is established.

In § 3 two consequences of this principle are discussed in order to gain a feeling for the scope and importance of the principle.

After having established the physical relevance of the concept of 'strictly larger mixing character' in § 2 and § 3, the subclass of mixing-homomorphic functionals, which preserve this relation, is investigated in § 4.1. A general criterion is obtained and it is shown that von Neumann's entropy and $-\mathrm{Tr}\,\rho^2$ have the required property of being strictly mixing homomorphic.

In § 4.2. the mentioned absolute quoset structure of observables is utilised in an interpolation theorem pointing to the fact that the 'larger mixing character' relation is most closely connected with or underlying the concept of entropy of observables with respect to a given density operator.

2. A generalisation of the Ruch-and-Mead principle

According to von Neumann (1955) the general directly measurable observable A has a purely discrete spectrum $\{a_n | n = 1, 2, \ldots\}$. In its spectral form: $A = \Sigma_n a_n P_n$ with $I = \Sigma_n P_n$, $P_n P_{n'} = \delta_{nn'} P_n$ and $(n \neq n' \Rightarrow a_n \neq a_{n'})$. If $\exists n$, $\mathrm{Tr}\, P_n > 1$, then A is *incomplete* and its measurement is an incomplete measurement; otherwise A and its measurement are complete.

As is well known, in the set of general observables of a quantal system, some pairs are related by a single-valued functional dependence: $A = f(B)$. It is, however, not widely realised that this defines a pre-order (or quasi-order) $A \leqslant B$ (a reflexive and transitive but not antisymmetric binary relation), making a quoset out of the set of general observables. The induced equivalence relation $(A \sim B \Leftrightarrow A \leqslant B$ and $B \leqslant A)$ holds if and only if A and B give (through their spectral forms) the same spectral decomposition of the identity. The induced partial ordering in the quotient poset is no other than the relation 'giving a finer or equal spectral decomposition of the identity'. Incomplete observables are the ones for each of which there exists an inequivalent upper bound: 'maximal' observables (in the sense of $\leqslant$) are the complete ones.

As further prerequisites for the generalisation, two results are required. The first is due to Lüders (1951, cf also Herbut 1969 and 1974). It says that the predictive non-selective measurement (i.e. the one the results of which apply to the immediate future and that refers to the entire ensemble (henceforth shortly: measurement)) of a general observable A converts an arbitrary density operator ρ into

$$\rho' \equiv \sum_n P_n \rho P_n. \tag{3}$$

The second result was proved in previous work (Herbut 1974, Theorem 2). For arbitrary A and ρ, there exists a complete observable $B = \Sigma_\alpha b_\alpha |\alpha\rangle\langle\alpha|$ $(\alpha \neq \alpha' \Rightarrow b_\alpha \neq b_{\alpha'})$ the measurement of which on ρ converts the latter into the same density operator as the measurement of A:

$$\rho' = \sum_n P_n \rho P_n = \sum_\alpha |\alpha\rangle\langle\alpha|\rho|\alpha\rangle\langle\alpha|. \tag{4}$$

Theorem 1. If ρ' is the density operator into which a general density operator ρ is converted by the measurement of a general observable A (cf (3)), then

$$m[\rho'] \geqslant m[\rho]. \tag{5}$$

The proof follows immediately from the preceding result, because for the complete observable B relation (5) has been proved by Ruch and Mead (1976). □

As pointed out in previous work (Herbut 1969, IV, A), one has

$$\rho' = \rho \Leftrightarrow [A, \rho] = 0; \tag{6}$$

in other words, the density operator changes if and only if the observable and the density operator are incompatible. As a consequence one has:

Theorem 2. The mixing character strictly increases in incompatible measurement:

$$[A, \rho] \neq 0 \Rightarrow m[\rho'] > m[\rho].$$

Proof. As evident from (4) and (6), $[A, \rho] \neq 0 \Leftrightarrow [B, \rho] \neq 0$, hence it is sufficient to treat complete observables. Now we repeat (and extend) the argument of Ruch and Mead (1976). Since we have to distinguish between matrices and operators, we write now (only in this proof) the density operator as $\hat{\rho}$.

Let ρ be the matrix that represents the initial density operator $\hat{\rho}$ in the eigenbasis $\{|\alpha\rangle|\forall\alpha\}$ of the complete observable B.

Let further $\tilde{\rho}$ be the diagonal representation of $\hat{\rho}$: $\tilde{\rho}_{\mu\nu}=p_\mu\delta_{\mu\nu}$. Then there exists a unitary matrix U such that $\rho=U\tilde{\rho}U^+$. This implies $\rho_{\alpha\alpha}\equiv p'_\alpha=\Sigma_{\mu\nu}\,U_{\alpha\mu}\tilde{\rho}_{\mu\nu}U^+_{\nu\alpha}=\Sigma_\mu\,|U_{\alpha\mu}|^2p_\mu$.

The measurement of B on $\hat{\rho}$ converts the latter into $\hat{\rho}'$. This operator is represented by ρ' in the eigenbasis of B and its elements are $\rho'_{\alpha\beta}=p'_\alpha\delta_{\alpha\beta}$. On the other hand, the elements $|U_{\alpha\mu}|^2$ make a so called bistochastic matrix, i.e. $\Sigma_\alpha\,|U_{\alpha\mu}|^2=\Sigma_\mu\,|U_{\alpha\mu}|^2=1$. Finally, there is a theorem of Hardy *et al* (1952) stating that $\forall\alpha$, $p'_\alpha=\Sigma_\mu\,|U_{\alpha\mu}|^2p_\mu$ amounts to the same as $\forall s$,

$$\sum_{j=1}^{s}p'_j\leqslant\sum_{j=1}^{s}p_j\quad\text{or}\quad m[\hat{\rho}']\geqslant m[\hat{\rho}].$$

This concludes the argument of Ruch and Mead.

Now, $[B,\hat{\rho}]\neq 0$ implies that ρ is not diagonal and hence the matrix $|U_{\alpha\mu}|^2$ is non-trivial ($|U_{\alpha\mu}|^2$ are not Kronecker symbols); and this in turn means that $m[\hat{\rho}']>m[\hat{\rho}]$. □

In view of the fact that in classical physics a statistical ensemble never changes in measurement, the above change from ρ to ρ' may be considered to be a fundamental quantal phnenomenon. It is accompanied by strict increase of mixing character.

3. Some consequences

3.1. Distant measurement and distant ensemble decomposition

Let the indices 1 and 2 refer to two particles that are now far apart and non-interacting, but they have interacted in the past and, owing to this, are now in a correlated pure state $|\phi_{12}\rangle$. Let us write it in its Schmidt canonical form (Herbut and Vujičić 1976, Theorem 4):

$$|\phi_{12}\rangle=\sum_m r_m^{1/2}|\varphi_m\rangle|\chi_m\rangle$$

where

$$\rho_1=\sum_m r_m|\varphi_m\rangle\langle\varphi_m|,\qquad \rho_2=\sum_m r_m|\chi_m\rangle\langle\chi_m|$$

are spectral forms of the reduced density operators

$$\rho_1\equiv\mathrm{Tr}_2\,|\phi_{12}\rangle\langle\phi_{12}|,\qquad \rho_2\equiv\mathrm{Tr}_1\,|\phi_{12}\rangle\langle\phi_{12}|$$

(Tr_1 and Tr_2 are the partial traces), and $\forall m$, $r_m>0$.

The measurement of a first-particle observable A_1 that is compatible with ρ_1 is simultaneously also a measurement on particle 2 though neither the measuring apparatus nor particle 1 are in interaction with particle 2. This was called distant measurement and studied in the mentioned previous work. Owing to this measurement $|\phi_{12}\rangle\langle\phi_{12}|$ changes into $\rho'_{12}=\Sigma_m\,r_m|\varphi_m\rangle\langle\varphi_m|\otimes|\chi_m\rangle\langle\chi_m|$, and hence

$$m[\rho'_{12}]>m[|\phi_{12}\rangle\langle\phi_{12}|],$$

but the reduced density operators are unchanged and hence so are their mixing characters.

When one measures a first-particle observable A_1 that is incompatible with the first-particle reduced density operator ρ_1 of the given correlated two-particle state vector $|\phi_{12}\rangle$: $[A_1, \rho_1] \neq 0$, then the resulting two-particle density operator is

$$\rho'_{12} = \sum_n |\alpha_1^{(n)}\rangle\langle\alpha_1^{(n)}|\phi_{12}\rangle\langle\phi_{12}|\alpha_1^{(n)}\rangle\langle\alpha_1^{(n)}|$$

or (7)

$$\rho'_{12} = \sum_n q_n|\alpha_1^{(n)}\rangle\langle\alpha_1^{(n)}|\otimes|\beta_2^{(n)}\rangle\langle\beta_2^{(n)}|.$$

Here $\Sigma_n\, a_n|\alpha_1^{(n)}\rangle\langle\alpha_1^{(n)}|\otimes I_2$ is the spectral form of A_1 (which is for simplicity taken to be complete for the first particle, but is incomplete for the two-particle system); $\forall n$, $q_n^{1/2}|\beta_2^{(n)}\rangle = \langle\alpha_1^{(n)}|\phi_{12}\rangle$ is the vector in the state space of the second particle obtained in the partial scalar product of the RHS (cf Herbut and Vujičić 1976, Appendix 1), the $|\beta_2^{(n)}\rangle$ are of norm 1 but non-orthogonal (ibid., Theorem 2), and $\forall n$, $q_n \equiv \langle\alpha_1^{(n)}|\rho_1|\alpha_1^{(n)}\rangle = \langle\phi_{12}|\alpha_1^{(n)}\rangle\langle\alpha_1^{(n)}|\phi_{12}\rangle$ is the probability to obtain a_n of A_1.

Both the two-particle density operator $|\phi_{12}\rangle\langle\phi_{12}|$ and its first-particle reduced density operator ρ_1 undergo a change in the measurement of the observable A_1 incompatible with them, and hence their mixing characters become strictly larger. The reduced density operator of the second particle ρ_2 does not change (since $[A_1, \rho_2] = 0$), and so neither does its mixing character. Nevertheless, the first-particle measurement at issue accomplishes something also on the second particles: it is distant non-orthogonal ensemble decomposition. Namely, when one evaluates $\rho_2 = \rho'_2 = \mathrm{Tr}_1\, \rho'_{12}$ from (7), one obtains $\rho_2 = \Sigma_n\, q_n|\beta_2^{(n)}\rangle\langle\beta_2^{(n)}|$ (not a spectral form). Generalising with Ruch (1975) the mixing character of density operators to those of classical discrete probability (or statistical weight) distributions in the obvious way (cf (1)), one may say that in the second-particle non-orthogonal ensemble decomposition under discussion, one has

$$m[q_n] > m[p_i]$$

(p_i being the eigenvalues of ρ_2). This is an immediate consequence of $m[\rho'_1] > m[\rho_1]$ and the fact that q_n are the eigenvalues of ρ'_1 and the p_i are those of ρ_1.

3.2. The mixing property of entropy and its counterparts in terms of mixing characters

The mixing property is an important feature of entropy and the latter may be derived from the former in the classical case (Shannon 1948 and Jaynes 1957a). To state it let us take a general observable $A = \Sigma_n\, a_n P_n (n \neq n' \Rightarrow a_n \neq a_{n'})$ in its spectral form such that $[A, \rho] = 0$. Then one can write (Herbut 1969, (A.10) and (A.11)):

$$\rho = \sum_n w_n \rho_n, \tag{8}$$

where $\forall n$, $w_n \equiv \mathrm{Tr}\, P_n\rho$ (the probabilty of a_n of A in ρ), and for $w_n > 0$: $\rho_n = w_n^{-1} P_n\rho$ (ρ_n gives a_n with certainty). For each n, $R(\rho_n)$ (the range of ρ_n) is a subspace of $R(P_n)$, hence the $R(\rho_n)$ are orthogonal. The mixing property consists in the following:

$$S(\rho) = S(w_n) + \sum_n w_n S(\rho_n) \tag{9a}$$

where

$$S(w_n) = -\sum_n w_n \ln w_n \tag{9b}$$

is the so-called mixing entropy.

An important special case of the mixing property is the so-called additivity of entropy: $S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$ (it follows from (9) when one takes e.g. ρ_1 in a spectral form $\rho_1 = \Sigma_i r_i |i\rangle\langle i|$).

Let us take for every $w_n > 0$, an arbitrary state vector $|\psi_n\rangle \in R(P_n)$ of norm 1. Defining

$$\bar{\rho} = \sum_n w_n |\psi_n\rangle\langle\psi_n| \tag{10}$$

(a spectral form), one has $S(\bar{\rho}) = S(w_n)$. Thus,

$$S(\rho) > S(\bar{\rho}) \tag{11}$$

if and only if, at least for one n, say n_0, $w_{n_0} > 0$ and ρ_{n_0} in (8) is mixed, i.e.

$$\rho^2_{n_0} \neq \rho_{n_0}. \tag{12}$$

This is where our problem begins. Comparing (8) plus (12) with (10), it is intuitively clear that ρ is more heterogeneous or more mixed than $\bar{\rho}$, and this is expressed by (11). Since the mixing character is a qualitative concept that seems to underly the quantitative concept of entropy, one intuitively expects (8) and (10) to imply

$$m[\rho] \geqslant m[\bar{\rho}], \tag{13a}$$

and if (12) holds, then

$$m[\rho] > m[\bar{\rho}]. \tag{13b}$$

Relations (13a) and (13b) are the *counterparts of the mixing property of entropy in terms of mixing characters.*

We prove now that (13a) (and (13b)) are indeed valid as formal consequences of the principle of increasing (strictly increasing) mixing character in general measurement.

Let

$$\rho = \sum_n w_n \sum_{t_n} r_{t_n} |t_n\rangle\langle t_n|$$

be a spectral form of ρ. Since (10) is a spectral form of $\bar{\rho}$, its mixing character depends only on the w_n, and not on the $|\psi_n\rangle$. Hence we may rechoose the latter as $|\psi'_n\rangle$ (preserving $|\psi'_n\rangle \in R(P_n)$). Defining $\bar{\bar{\rho}} \equiv \Sigma_n w_n |\psi'_n\rangle\langle\psi'_n|$, we have $m[\bar{\bar{\rho}}] = m[\bar{\rho}]$. Let $|\psi'_n\rangle \equiv \Sigma_{t_n} r^{1/2}_{t_n} |t_n\rangle$; and let us define $B \equiv \Sigma_{n(w_n>0)} \Sigma_{t_n} b_{n,t_n} |t_n\rangle\langle t_n + |\ldots$ as a complete observable in its spectral form. Its measurement converts $\bar{\bar{\rho}}$ into $\Sigma_n w_n \Sigma_{t_n} r_{t_n} |t_n\rangle\langle t_n| = \rho$. If (12) is valid, then t_{n_0} takes on more than one value. Then $[|\psi'_{n_0}\rangle\langle\psi'_{n_0}|, |t_{n_0}\rangle\langle t_{n_0}|] \neq 0$ (with any one of these values t_{n_0}). Since all $|\psi'_n\rangle$, $n \neq n_0$, are orthogonal to $|t_{n_0}\rangle \in R(P_{n_0})$,

$$[\bar{\bar{\rho}}, |t_{n_0}\rangle\langle t_{n_0}|] = w_{n_0}[|\psi'_{n_0}\rangle\langle\psi'_{n_0}|, |t_{n_0}\rangle\langle t_{n_0}|] \neq 0.$$

In view of the fact that $\bar{\bar{\rho}}$ commutes with B if and only if the former commutes with each eigenprojector of the latter, we conclude that $[\bar{\bar{\rho}}, B] \neq 0$. The principle of strictly increasing mixing character in incompatible measurement then finally gives (13b).

The weaker relation (13*a*) is an immediate consequence of the fact that $\hat{\rho}$ goes over into ρ by measurement. This concludes the proof.

4. Entropy and other mixing-homomorphic functionals

4.1. *Strictly mixing-homomorphic functionals*

Having discussed some consequences of the binary relation 'having a strictly larger mixing character' in the set of density operators, we proceed to discuss functionals which preserve this relation, i.e., which are strictly mixing homomorphic.

To define a mixing-homomorphic functional on the set of all density operators ρ, Ruch and Mead (1976) consider functionals of the form $G(g,\rho)\equiv \mathrm{Tr}\, g(\rho)=\Sigma_\alpha\, g(p_\alpha)$, where $\{p_\alpha|\forall_\alpha\}$ is the spectrum of ρ (with repetitions of the eigenvalues in the case of multiplicities), and g is some suitable real function. By definition $G(g,\rho)$ is mixing homomorphic if

$$(m[\rho']\geqslant m[\rho])\Rightarrow[G(g,\rho')\geqslant G(g,\rho)]. \tag{14}$$

If both inequalities in (14) are strict, we say that $G(g,\rho)$ is *strictly mixing homomorphic.* As can be easily seen, this relation is stronger than 'mixing homomorphic'.

The function $g(x)$ is strictly concave in $(0, 1)$ if for every choice of $x_\alpha\in(0,1)$, $c_1>0<c_2$, $c_\alpha\geqslant 0$, $\alpha=3,4,\ldots,\Sigma_\alpha\, c_\alpha=1$, one has

$$g\Big(\sum_\alpha c_\alpha x_\alpha\Big)>\sum_\alpha c_\alpha g(x_\alpha). \tag{15}$$

(Ruch and Mead 1976 would call this function strictly convex; our terminology is in agreement with that of Wehrl 1978). If $g(x)$ is twice differentiable, then $\mathrm{d}^2g(x)/\mathrm{d}x^2<0$ in $(0, 1)$ is a necessary and sufficient condition for strict concavity.

Theorem 3. If $g(x)$ is strictly concave in $(0, 1)$, then $G(g,\rho)$ is a strictly mixing homomorphic functional.

Proof. If $m[\rho']>m[\rho]$, then there exists a non-trivial bistochastic matrix $(B_{\alpha\beta})$ such that $\forall\alpha$, $p'_\alpha=\Sigma_\beta\, B_{\alpha\beta}p_\beta$, where $\{p'_\alpha|\forall\alpha\}$ and $\{p_\beta|\forall\beta\}$ are the spectra of ρ' and ρ respectively. Inequality (15) implies: $\forall\alpha$, $g(p'_\alpha)>\Sigma_\beta\, B_{\alpha\beta}g(p_\beta)$. Summing over α, one arrives at $\Sigma_\alpha\, g(p'_\alpha)\equiv G(g,\rho')>\Sigma_\beta\, g(p_\beta)\equiv G(g,\rho)$. □

It is now easy to see that $-\|\rho\|_k\equiv-(\Sigma_\beta\, p_\beta^k)^{1/k}$ and the con Neumann entropy $S(\rho)\equiv-\Sigma_\beta\, p_\beta \ln p_\beta$ are strictly mixing homomorphic.

As mentioned previously, the von Neumann entropy has the property of additivity and hence it is irreplaceable for composite physical systems. But for non-composite ones a simpler functional, say $-\|\rho\|_2=-(\mathrm{Tr}\,\rho^2)^{1/2}$, is just as good for the purpose of making the set of all ρ's totally pre-ordered via a strictly mixing-homomorphic functional.

4.2. *Mixing character and entropy in relation to the quoset of observables*

As it was explained at the beginning of § 2, the set of general observables of a given quantal system is a quoset with $A\leqslant B\Leftrightarrow A=f(B)$ as the quasi-order defining this structure. When an arbitrary density operator ρ is given, then the relation 'having a larger mixing character' among the probability distributions on the spectra of the observables in the state ρ introduces a new quoset structure in the set of all observables. The latter is relative: it depends on the selection of ρ.

Theorem 4. When two observables $A = \Sigma_n a_n P_n$ $(n \neq n' \Rightarrow a_n \neq a_{n'})$ and $B = \Sigma_m b_m Q_m$ $(m \neq m' \Rightarrow b_m \neq b_{m'})$ are in the absolute quasi-order relation $A \leqslant B$, then also $\forall \rho$, $m[p(a_n|\rho)] \leqslant m[p(b_m|\rho)]$. This homomorphism is in general not strict, i.e. one may have $A < B$ and $m[p(a_n|\rho)] = m[p(b_m|\rho)]$.

Proof. The relation $A \leqslant B$ enables one to rewrite B in the spectral form $B = \Sigma_n \Sigma_{m_n} b_{n,m_n} Q_{n,m_n}$ so that $\forall n, \Sigma_{m_n} Q_{n,m_n} = P_n$. Hence $p(a_n|\rho) \equiv \mathrm{Tr}\, P_n\rho = \Sigma_{m_n} p(b_{n,m_n}|\rho)$. Now we can construct $\rho' \equiv \Sigma_n p(a_n|\rho) \Sigma_{m_n} [p(b_{n,m_n}|\rho)/p(a_n|\rho)] |\psi_{n,m_n}\rangle\langle\psi_{n,m_n}|$ with any $|\psi_{n,m_n}\rangle \in R(Q_{n,m_n})$ achieving (8) for ρ', and further $\bar{\rho} \equiv \Sigma_n p(a_n|\rho)|\psi_n\rangle\langle\psi_n|$ with any $|\psi_n\rangle \in R(P_n)$. Relation (13*a*) then entails the first claim of theorem 4. To prove the second claim, let us take in the two-dimensional spin space: $A \equiv |+\rangle\langle+|+|-\rangle\langle-|$, $B \equiv |+\rangle\langle+|-|-\rangle\langle-|$, and $\rho \equiv |+\rangle\langle+|$. Then $A < B$, and $m[p(a_n|\rho)] = m[p(b_{n,m_n}|\rho)]$. □

Corollary. The relation $A \leqslant B$ implies $S[p(a_n|\rho)] \leqslant S[p(b_m|\rho)]$, and this homomorphism is not strict. The same is true for every other mixing-homomorphic functional as well.

Thus, in the set of general observables one has, besides the absolute quoset structure, also the quoset structure due to the relation '$\succcurlyeq$' or 'having a larger mixing character in ρ' and the total pre-order 'having larger entropy in ρ'. The relative quasi-order '$\succcurlyeq$' is interpolated between the absolute quasi-order and the total pre-order of the entropy, being 'closer' to the latter because it is from '$\succcurlyeq$' to 'having larger entropy in ρ' where the strict homomorphism holds. In this sense '$\succcurlyeq$' underlies entropy (cf also § 3.2).

It is noteworthy that two observables A_1 and A_2 have a common upper bound B: $A_1 \leqslant B$, $A_2 \leqslant B$, if and only if $[A_1, A_2] = 0$ (cf von Neumann 1955, p 173). Clearly, with relation to a density operator ρ, a common upper bound can appear also for two incompatible observables. For instance, let B and $\bar{B}$ be two complete observables, the former compatible and the latter incompatible with ρ, then $m[p(b_\beta|\rho)] < m[p(\bar{b}_\beta|\rho)]$ though B and $\bar{B}$ are incomparable regarding the absolute pre-order and are incompatible with each other. It is due to the fact that $\{p(b_\beta|\rho)|\forall\beta\}$ is the spectrum of ρ and that for the measurement of $\bar{B}$ the principle of strictly increasing mixing character is valid.

5. Summary

The principle of increasing mixing character in general measurement is shown to be valid. The principle of strictly increasing mixing character in incompatible general measurement and its consequences give a clear and sufficiently important physical meaning to the binary relation 'strictly larger' in the lattice of mixing characters.

The functionals $G(g, \rho)$ based on a strictly concave function $g(x)$ preserve the 'strictly larger' relation. The von Neumann entropy belongs to this class.

Finally, the relative quoset structure of general observables with respect to any density operator is interpolated between the absolute quoset structure and the totally quasi-ordered set obtained via a strictly mixing-homomorphic functional.

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